NOTE ON CONVERGENCE TESTS FOR SERIES AND ON STIELTJES INTEGRATION BY PARTS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society September 4, 1919.)

THE obvious identity

$$(1)\qquad [c_k^{(1)}c_k^{(2)}\cdots c_k^{(n)} - a_k^{(1)}a_k^{(2)}\cdots a_k^{(n)}] = [a_k^{(2)}a_k^{(3)}\cdots a_k^{(n)}(c_k^{(1)} - a_k^{(1)})] + \sum_{i=2}^{n-1}[c_k^{(1)}\cdots c_k^{(i-1)}a_k^{(i+1)}\cdots a_k^{(n)}(c_k^{(i)} - a_k^{(i)})] + [c_k^{(1)}\cdots c_k^{(n-1)}(c_k^{(n)} - a_k^{(n)})]$$

may serve in several ways for the investigation of the convergence of series. In each it is to be used as a means of relating the $n+1$ sums of the first m terms of $n+1$ different series the general kth terms of which in order are the $n+1$ bracketed expressions in the foregoing identity. *It is obvious that the convergence of any n of these series implies the convergence of the remaining one.* Moreover, when they all converge, the identity leads to an obvious relation among the $n+1$ sums.

The effectiveness of the whole class of theorems arising thus lies in the fact that the series whose convergence is asserted may be, and in important cases is, conditionally convergent while the auxiliary series are absolutely convergent.

To arrive at a special instance of the theorems thus indicated, let us put $a_k^{(i)} = c_{k-1}^{(i)}$ with $c_0^{(i)} = 0$ and sum in (1) with respect to k from 1 to m. Then we have

$$(2)\qquad \begin{aligned} c_m^{(1)}c_m^{(2)}\cdots c_m^{(n)} &= \sum_{k=1}^{m} c_{k-1}^{(2)}c_{k-1}^{(3)}\cdots c_{k-1}^{(n)}(c_k^{(1)} - c_{k-1}^{(1)}) \\ &+ \sum_{i=2}^{n-1}\sum_{k=1}^{m} c_k^{(1)}\cdots c_k^{(i-1)}c_{k-1}^{(i+1)}\cdots c_{k-1}^{(n)} \\ &\times (c_k^{(i)} - c_{k-1}^{(i)}) + \sum_{k=1}^{m} c_k^{(1)}\cdots c_k^{(n-1)}(c_k^{(n)} - c_{k-1}^{(n)}). \end{aligned}$$

From this relation it is easy to see that we have the following result:*

* The theorems of the paper are given in forms suitable for ready use where applicable. In view of the general remark in the first paragraph it is clear that stronger (though less simply stated) theorems may be obtained from identity (2). A similar remark may be made about other results in the paper.

THEOREM I. *The convergence of the series*

$$\sum_{k=1}^{\infty} c_{k-1}{}^{(2)} c_{k-1}{}^{(3)} \cdots c_{k-1}{}^{(n)} (c_k{}^{(1)} - c_{k-1}{}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} | c_k{}^{(i)} - c_{k-1}{}^{(i)} |, \quad (i = 2, 3, \cdots, n),$$

and the existence of the limits

$$\lim_{k=\infty} c_k{}^{(1)} c_k{}^{(2)} \cdots c_k{}^{(n-1)},$$

$$\lim_{k=\infty} c_k{}^{(1)} \cdots c_k{}^{(i-1)} c_{k-1}{}^{(i+1)} \cdots c_{k-1}{}^{(n)}, \ (i = 2, \cdots, n-1).$$

In the special case when $n = 2$ let $c_k{}^{(1)}$ be the sum of the first $k-1$ terms of the converging infinite series $\alpha_1 + \alpha_2 + \cdots$, so that $c_k{}^{(1)} - c_{k-1}{}^{(1)} = \alpha_{k-1}$. Then we have the following theorem: *If the series* $\Sigma_k \alpha_k$ *and* $\Sigma_k | c_k{}^{(2)} - c_{k-1}{}^{(2)} |$ *converge, so does also the series* $\Sigma_k \alpha_k c_k{}^{(2)}$. Owing to its wide and fundamental use in the theory of Dirichlet series, factorial series, and more general classes of series, this special case of our Theorem I must now be considered one of the classic tools of mathematical analysis.

The same identity (2) can be made to yield another general theorem of which the instance $n = 2$ is also classic. It may be stated as follows:

THEOREM II. *The convergence of the series*

$$\sum_{k=1}^{\infty} c_{k-1}{}^{(2)} c_{k-1}{}^{(3)} \cdots c_{k-1}{}^{(n)} (c_k{}^{(1)} - c_{k-1}{}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} | c_k{}^{(i)} - c_{k-1}{}^{(i)} |, \qquad (i = 2, 3, \cdots, n),$$

the existence of the limit $\lim_{k=\infty} c_k{}^{(1)} c_k{}^{(2)} \cdots c_k{}^{(n)}$, *and the existence of a constant* B *(independent of* k *and* i*) such that for every* k *we have*

$$| c_k{}^{(1)} c_k{}^{(2)} \cdots c_k{}^{(n-1)} | < B, \quad | c_k{}^{(1)} \cdots c_k{}^{(i-1)} c_{k-1}{}^{(i+1)} \cdots c_{k-1}{}^{(n)} | < B \text{ for } i = 2, 3, \cdots, n.$$

On taking $c_k{}^{(i)} = c_k$ for $i = 2, 3, \cdots, n$, and writing ν for

$n-1$, we have the following special case which deserves separate statement:

The convergence of the series

$$\sum_{k=1}^{\infty}(c_{k-1})^{\nu}(c_k^{(1)}-c_{k-1}^{(1)})$$

is implied by the convergence of the series $\Sigma_k|c_k-c_{k-1}|$ *and either of the two following conditions*:

(1) *the existence of the limits*

$$\lim_{k=\infty} c_k^{(1)}(c_k)^{\nu-i}(c_{k-1})^{i-1}, \qquad (i=1,2,\cdots,\nu);$$

(2) *the existence of the limit* $\lim_{k=\infty} c_k^{(1)}(c_{k-1})^{\nu}$ *and the existence of a constant* B *(independent of* k *and* i*) such that for every* k *we have*

$$|c_k^{(1)}(c_k)^{\nu-i}(c_{k-1})^{i-1}|<B, \qquad (i=1,2,\cdots,\nu).$$

If the quantities $c_k^{(i)}$ are not constants but are functions of a variable x on a linear interval or in a domain of the complex plane, then it is clear that the series whose convergence is asserted in these theorems will converge uniformly if the conditions in the hypotheses are satisfied uniformly. This means, in Theorem I, that the limits explicitly mentioned and those implied by the convergent series in the hypothesis shall exist uniformly. In the case of Theorem II it means that the limits stated or implied in the hypothesis shall exist uniformly and that the inequalities shall be valid for a B independent not only of k and i but also of x. The instance $n=2$ of each of these theorems is now also to be considered classic.

If in (1) we set the a's equal to unity and in the resulting identity sum as to k from 1 to m, we have

$$\sum_{k=1}^{m}(c_k^{(1)}c_k^{(2)}\cdots c_k^{(n)}-1)=\sum_{k=1}^{m}(c_k^{(1)}-1) + \sum_{i=2}^{n}\sum_{k=1}^{m}c_k^{(1)}c_k^{(2)}\cdots c_k^{(i-1)}(c_k^{(i)}-1). \tag{3}$$

From this we obtain the following theorem:

THEOREM III. *Suppose that the series*

$$\sum_{k=1}^{\infty}|c_k^{(i)}-1|, \qquad (i=2,3,\cdots,n),$$

converge; then if either of the series

$$\sum_{k=1}^{\infty}(c_k^{(1)}c_k^{(2)}\cdots c_k^{(n)}-1) \quad \textit{and} \quad \sum_{k=1}^{\infty}(c_k^{(1)}-1)$$

converges, the other does also.

A similar theorem for uniform convergence may be formulated in an obvious manner.

If we examine the first two of the foregoing theorems in relation to identity (2), we see that the hypothesis in each case is framed so as to secure simply the existence of the limit $\lim_{k=\infty} c_k^{(1)}c_k^{(2)}\cdots c_k^{(n)}$ and the convergence of the series

$$(4)\qquad \begin{gathered}\sum_{k=1}^{\infty}c_k^{(1)}\cdots c_k^{(n-1)}(c_k^{(n)}-c_{k-1}^{(n)}),\\ \sum_{k=1}^{\infty}c_k^{(1)}\cdots c_k^{(i-1)}c_{k-1}^{(i+1)}\cdots c_{k-1}^{(n)}(c_k^{(i)}-c_{k-1}^{(i)}),\\ (i=2,3,\cdots,n-1);\end{gathered}$$

moreover, the existence of this limit and the convergence of these series are sufficient to secure the convergence of the series whose convergence is asserted. Now it will be observed that the series (4) are separately of the same form as that whose convergence is asserted in the first two theorems. Hence these theorems themselves may be used for testing the convergence of series (4); and similarly for the auxiliary series employed in applying these theorems to series (4). Thus we have a sequence of theorems for testing convergence. To state them in general in terms of a parameter would call for complicated formulas; these will not be given, since it is probably just as expeditious to use the given theorems repeatedly. For the case $n = 2$, however, the generalizations may be simply stated; these and others have been given in this BULLETIN, volume 20 (1914), pages 225–233 and in *Tôhoku Mathematical Journal*, volume 11 (1917), pages 191–199.

I was led to a consideration of identity (1) in connection with the problem of integration of a Stieltjes integral by parts. It is of interest to treat that problem in connection with the foregoing, particularly since it may be used to illustrate the fact that a generalization of Abel's transformation of series is afforded by identity (1).

Let π be a partition of the interval (ab) of norm δ due to the points $x_0 = a, x_1, x_2, \cdots, x_{n-1}, x_n = b$ and let ξ_i be a point of (x_{i-1}, x_i). We have the identity

$$\sum_{i=1}^{n} f(\xi_i)\{v_1(x_i)v_2(x_i) - v_1(x_{i-1})v_2(x_{i-1})\} \equiv \sum_{i=1}^{n} f(\xi_i)v_1(x_i)[v_2(x_i) - v_2(x_{i-1})] + \sum_{i=1}^{n} f(\xi_i)v_2(x_{i-1})[v_1(x_i) - v_1(x_{i-1})].$$

If we take $\xi_i = x_i$ and let δ approach zero, this relation yields the equation

$$\int_a^b f(x)d\{v_1(x)v_2(x)\} = \int_a^b f(x)v_1(x)dv_2(x) + \int_a^b f(x)v_2(x)dv_1(x), \tag{5}$$

provided that these Stieltjes integrals exist and that

$$\lim_{\delta=0} \sum_{i=1}^{n} v_2(x_{i-1})[f(x_i) - f(x_{i-1})][v_1(x_i) - v_1(x_{i-1})] = 0. \tag{6}$$

By taking $\xi_i = x_{i-1}$ and letting δ approach zero we have the same relation (5) provided that the integrals in it exist and that

$$\lim_{\delta=0} \sum_{i=1}^{n} v_1(x_i)[f(x_i) - f(x_{i-1})][v_2(x_i) - v_2(x_{i-1})] = 0. \tag{7}$$

Now if $f(x)$ is continuous on (ab) and $v_1(x)$ $[v_2(x)]$ is of bounded variation on (ab) while $v_2(x)$ $[v_1(x)]$ is bounded, it is obvious that (6) [(7)] is a valid relation. Thus we have sufficient conditions for the validity of (5) in case the integrals in (5) exist.

In particular, if $f(x)$ and either $v_1(x)$ or $v_2(x)$ are continuous on (ab) and both $v_1(x)$ and $v_2(x)$ are of bounded variation on (ab), then the integrals in (5) certainly exist and that relation is a valid one.

Let $f(x)$ be identically equal to 1 and write $g(x)$ for $v_1(x)$ and $v(x)$ for $v_2(x)$. Then on interchanging the members in (5) we have

$$\int_a^b g(x)dv(x) + \int_a^b v(x)dg(x) = g(b)v(b) - g(a)v(a), \tag{8}$$

provided these integrals exist. This special case of (5) is one of the frequently used formulas in the theory of Stieltjes integration; it is a generalization of the theorem for integration by parts in the theory of Cauchy-Riemann integration.

The identity just preceding relation (5) is a generalization of that involved in the classic transformation of Abel; for, if we replace $v_1(x)$ by $g(x)$, $v_2(x)$ by $v(x)$, and $f(x)$ by 1, we have the Abel identity

$$\sum_{i=1}^{n} g(x_i)[v(x_i) - v(x_{i-1})] \equiv -\sum_{i=1}^{n} v(x_{i-1})[g(x_i) - g(x_{i-1})] + g(b)v(b) - g(a)v(a).$$

It is obvious that a repeated use of (5) reduces the integral of $f(x)$ as to a product $v_1(x)v_2(x) \cdots v_n(x)$ to a sum of n integrals of functions as to $v_1(x)$, $v_2(x)$, $\cdots$, $v_n(x)$, respectively, under appropriate conditions like (6) and (7) and the hypothesis of the existence of these n integrals. The question arises naturally as to whether some simple identity exists, analogous to that employed in deriving (5), which would yield the entire result at once. It was through this question that I was led to identity (1). Having it, it is natural to extend the classic theorems about convergence of series previously obtained through the particular case which yields Abel's transformation. The reader will have no difficulty in obtaining through identity (1) the transformation of a Stieltjes integral mentioned at the beginning of this paragraph.

UNIVERSITY OF ILLINOIS,
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NOTE ON A PHYSICAL INTERPRETATION OF STIELTJES INTEGRALS.

BY PROFESSOR R. D. CARMICHAEL.

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STIELTJES was led to his definition of integral by what he called the problem of moments (see §24 of his memoir in *Annales de la Faculté des Sciences de Toulouse*, 1894). Consider on a straight line OX a distribution of (positive) mass, the mass m_i being concentrated at the distance ξ_i from the origin O. The sum $\Sigma m_i \xi_i^k$ be called the moment of order k of the mass with respect to the origin. He also considered the more general distribution of mass on OX which is such that

the mass on the segment $(0x)$ from 0 to x is $\varphi(x)$ where $\varphi(x)$ is a non-decreasing function which is finite for every finite value of x. Let us consider the moment about O, of order k, of the mass so distributed on the interval (ab) where a and b are both positive. Let π be a partition of (ab) of norm δ due to the points $x_0 = a, x_1, \cdots, x_{n-1}, x_n = b$, and let ξ_i be a point of the interval (x_{i-1}, x_i). Then for the moment in question we should obviously have the value

$$\lim_{\delta=0} \sum_{i=1}^{n} \xi_i^k \{\varphi(x_i) - \varphi(x_{i-1})\}.$$

That is, the value of the moment of order k is expressed by the Stieltjes integral

$$\int_a^b x^k d\varphi(x). \tag{1}$$

We thus have Stieltjes' physical interpretation of the integral (1) for each value of k, the function $\varphi(x)$ being monotonic.

The object of this note is to throw the physical interpretation of this integral into another form and to generalize the new form so as to afford a physical interpretation for the integral of $f(x)$ as to any function $v(x)$ of bounded variation.

Let us consider the curve $y = x^k$ and a distribution of mass on that curve such that the mass on the segment from (0, 0) to (x, y) shall be $\varphi(x)$. Then the integral (1) represents the usual (first) moment of the mass on the segment of $y = x^k$ from $x = a$ to $x = b$ with respect to the axis of x, as one sees readily from the foregoing finite sum of which the integral (1) is the limit.

Now this interpretation of integral (1) admits of an obvious generalization. Let $f(x)$ be any single-valued function of x which is bounded on (ab) and let positive mass be distributed along the curve $y = f(x)$ so that the mass on the segment from $x = a$ to $x = x$ shall be $\varphi(x)$, where the functions $f(x)$ and $\varphi(x)$ have no common singularity and where the total mass involved is finite. Then the integral

$$\int_a^b f(x) d\varphi(x)$$

exists and gives the value of the moment about the x-axis of the mass on the curve $y = f(x)$ from $x = a$ to $x = b$.

Thus we have a physical interpretation for the integral of a bounded function with respect to any bounded monotonic non-decreasing function. If we avail ourselves of the notion of negative mass we may obtain similarly a physical interpretation for the integral of a bounded function $f(x)$ as to a function $v(x)$ of bounded variation.

Write $v(x)$ in the form

$$v(x) = v(a) + P(x) - N(x),$$

where $P(x)$ and $N(x)$ are respectively the positive and the negative variation of $v(x)$ on (ax). Then along the curve $y = f(x)$ let us have a distribution of positive mass such that the positive mass on the segment from $x = a$ to $x = x$ is $P(x)$; along the same curve let us have a distribution of negative mass such that the negative mass on the segment from $x = a$ to $x = x$ is $N(x)$. We suppose that neither $P(x)$ nor $N(x)$ is discontinuous at a discontinuity of $f(x)$. Then the moment of the positive mass about the axis will be represented by the integral of $f(x)$ as to $P(x)$; that of the negative mass by the negative of the integral of $f(x)$ as to $N(x)$. Hence the algebraic sum of these moments, or the total moment about the x-axis, is represented by

$$\int_a^b f(x)dv(x). \tag{2}$$

Such a physical interpretation of the integral is useful in giving one a better intuitive sense of its character and hence in affording a means of classifying its properties in his thought.

If one is disturbed by the notion of negative mass which enters into the foregoing interpretation he may avoid its use by resorting to another range of physical phenomena. He may replace positive and negative mass by positive and negative magnetism. Then $P(x)$ $[N(x)]$ would denote the total positive [negative] magnetism on the segment of $y = f(x)$ from $x = a$ to $x = x$ and the moment would be taken with respect to the x-axis under a magnetic field of force of unit intensity in a direction perpendicular to the xy-plane. The moment will again be represented by the integral (2).

It is obvious that this representation may also be modified into a geometrical one. Consider the cylindrical surface perpendicular to the xy-plane and intersecting it in the curve

$y = f(x)$. On the positive side of the xy-plane take the curve lying on this cylindrical surface and satisfying the condition $z = P(x)$; on the other side take the curve lying on the cylindrical surface and satisfying the condition $z = -N(x)$. We thus intercept between two curves a part of the cylindrical surface for which x is on (ab). It is not difficult to form a fair intuitive notion of this portion of the surface since $P(x)$ and $N(x)$ are both monotonic non-decreasing. Then the integral (2) is the "area" of the part of this bounded cylindrical surface lying on the positive side of the xy-plane minus the "area" of that part lying on the other side.

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A DERIVATION OF THE EQUATION OF THE NORMAL PROBABILITY CURVE.

BY PROFESSOR W. D. CAIRNS.

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THE symmetrical distribution of magnitudes about their mean is commonly represented by a "polygon" whose equally spaced ordinates are proportional to the terms of the expansion of $(1 + 1)^n$. The statement is frequently made in textbooks without any proof that as n is increased indefinitely, the equal spaces and the vertical scale being properly controlled, the polygon approaches as its limiting form the normal curve $y = ke^{-h^2x^2}$. The method here given for the proof of this theorem consists essentially in controlling what may be called the points of inflexion of the polygon so that these points approach predetermined positions on each side of the mean. Since an extended and rigorous proof of the probability theorem has been published,* it will suffice to indicate here the general plan of the proof.

* E. L. Dodd, *American Mathematical Monthly*, vol. 20 (1913), p. 128. Since this paper was written, a proof by A. A. Bennett has appeared in this BULLETIN, volume 24, No. 10, page 477. In that proof the area (rather than the standard deviation, as used in the present paper) and the middle ordinate control the curve and Wallis's product formula for $\pi/2$ is used. Since the area and the standard deviation are alike fundamental in the applications to probability, statistics, theory of errors, etc., it would seem that each of these gives a natural method of approach.

By the elementary binomial formula the middle (greatest) term of $(1+1)^{2n}$ is $(2n)!/(n!)^2$; and by the shortened Stirling's formula

$$n! \sim \sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}$$

this term approximates for large values of n to $2^{2n}/\sqrt{\pi n}$. If then we assign the terms of $\frac{\sqrt{\pi n}}{2^{2n}}(1+1)^{2n}$ to points on the x axis to the left and right of the origin at distances Δx apart, the middle term, assigned to the origin, approximates for large values of n to unity (and thus, by the introduction of a constant factor, to any preassigned value).

The kth term to the right (or left) is

$$\frac{\sqrt{\pi n}}{2^{2n}} \frac{(2n)!}{(n+k)!(n-k)!}$$

and this by Stirling's formula approximates to

$$\frac{\sqrt{n}}{2^{2n}} \frac{(2n)^{2n+\frac{1}{2}}}{\sqrt{2}(n+k)^{n+k+\frac{1}{2}}(n-k)^{n-k+\frac{1}{2}}},$$

i.e., to

$$\left(\frac{n-k}{n+k}\right)^k \Big/ \left(1-\frac{k^2}{n^2}\right)^{n+\frac{1}{2}},$$

the form here used for our ordinate.

Aside from the factor $\sqrt{\pi n}(2n)!/2^{2n}$, the difference between the kth and the $(k+1)$th term is

$$\frac{2k+1}{(n+k+1)!\,(n-k)!}.$$

Hence the k_σth term for which the ordinates of the polygon decrease most rapidly is characterized by the condition

$$\frac{2k_\sigma - 1}{(n+k_\sigma)!\,(n-k_\sigma+1)!} \leqq \frac{2k_\sigma+1}{(n+k_\sigma+1)!\,(n-k_\sigma)!} > \frac{2k_\sigma+3}{(n+k_\sigma+2)!\,(n-k_\sigma-1)!},$$

or by an easy reduction

$$k_\sigma \leqq \sqrt{(n+1)/2} < k_\sigma + 1.$$

If therefore as n increases we choose Δx so that $\sqrt{n/2}\cdot\Delta x\cdot$ equals σ, a predetermined positive constant, the last condition may be written

$$k_\sigma \Delta x \leqq \sigma\sqrt{(n+1)/n} < k_\sigma \Delta x + \Delta x;$$

hence the k_σth term will with increasing n approach the position whose abscissa is σ. If further we set $k\Delta x = x$, i.e.,

$$k = \frac{x}{\sigma}\sqrt{n/2},$$

the kth ordinate, assigned to the position whose abscissa is x, will be

$$\frac{1}{\left(1 \vdash \frac{x^2}{2\sigma^2 n}\right)^{n+\frac{1}{2}}} \left(\frac{n - \frac{x}{\sigma}\sqrt{\frac{n}{2}}}{n + \frac{x}{\sigma}\sqrt{\frac{n}{2}}}\right)^{\frac{x}{\sigma}\sqrt{\frac{n}{2}}}.$$

Finally, by evaluating this expression for $n = \infty$, it is found to approach uniformly the limit $e^{-x^2/2\sigma^2}$, whence the equation of the limiting form of curve is $y = e^{-x^2/2\sigma^2}$, the customary form of the normal curve equation save for an arbitrary constant factor.

From the foregoing it also follows that if instead of each ordinate there be used a rectangle of the same height and of base Δx, the sum of the areas of the rectangles approximates asymptotically to $\sqrt{\pi n}\Delta x$, i.e., to $\sigma\sqrt{2\pi}$; in other words, (1) the area under the curve is finite and (2) we have evaluated the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sigma\sqrt{2\pi}.$$

The writer has derived a series of formulas, a representative of which is the following:

$$\sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!\,(n-k)!} k^2 = \frac{n}{2} 2^{2n};$$

i.e., if the binomial coefficients (or terms) of $(1+1)^{2n}$ be multiplied respectively by the squares of the number of the terms counting from the midterm, the sum of the products is as stated. (The theorem may also be stated in terms of

mean square deviation.) The use of this formula in the foregoing method gives the result

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^2 dx = \lim_{n=\infty} \sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!\,(n-k)!} (k\Delta x)^2 \Delta x = \sigma^3 \sqrt{2\pi}.$$

Similar evaluations are obtained for

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^4 dx, \qquad \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^6 dx, \text{ etc.}$$

OBERLIN, OHIO.

BÔCHER'S BOUNDARY PROBLEMS FOR DIFFERENTIAL EQUATIONS.

Leçons sur les Méthodes de Sturm dans la Théorie des Équations Différentielles Linéaires et leurs Développements Modernes, professées à la Sorbonne en 1913–1914. Par MAXIME BÔCHER. Recueillies et rédigées par GASTON JULIA. Paris, Gauthier-Villars, 1917. 6 + 118 pp.

IT can be said without fear of contradiction that what may be characterized as the *linear problem* is one of the most central in all mathematics. In algebra this problem concerns itself not only with linear forms and linear equations but also with many phases of the discussion of bilinear and quadratic forms. The results arrived at from an algebraic treatment find immediate application in geometry and mechanics. In the field of analysis the linear differential equation in one or more independent variables has always occupied a position of prime importance and in recent years the study of linear integral equations has not only forged a new and powerful tool but has also exerted a profound influence on the general trend of mathematical thought. The recent development of the theory of linear algebraic equations in an infinite number of unknowns by bridging the gap between the old algebraic field of linear equations and bilinear forms on the one hand and the analytic field of differential equations, integral equations, and bilinear forms in an infinite number of variables on the other, has given a remarkable unity to the various aspects of the general problem. In searching for the theory

underlying the various forms of the linear problem, Moore has been led to the development of his General Analysis.

Perhaps the most interesting questions of relative maximum and minimum and of calculus of variations are those intimately bound up with the solutions of problems arising in these other domains. The linear differential equation as well as the linear algebraic equation owes much of its importance to its connection with applied mathematics. In fact the original discussion by Sturm was suggested by problems arising in mathematical physics, and Bôcher himself, while a student at Göttingen in the late eighties and early nineties, was led into this field by problems of a like origin which were interesting Klein and his school.

Among the methods employed during recent years in the study of boundary problems, those of asymptotic expression and successive approximations pertain more or less strictly to the field of differential equations itself. It is, however, not surprising that other disciplines have been called in to aid in the development of the theory. Among the newer tools to be used in the attack might be included integral equations, calculus of variations, passage to the limit from a finite number of approximating linear equations in a finite number of variables, and linear equations in an infinity of variables. It was, however, one of the theses stoutly defended by Bôcher that the newer methods are more artificial and less elegant than those of Sturm and that they add little to the theory.* While admitting to some extent the force of this argument, it should be pointed out that the analogies suggested by the fresh points of view have been fruitful in indicating new problems and new avenues of advance even by the old methods. And it is especially worthy of note that while the Sturmian methods proved incapable of solving boundary problems in two or more dimensions, the others lend themselves readily to such generalization.

In his article in the Encyklopädie der Mathematischen Wissenschaften, Professor Bôcher summarizes the main results attained in the discussion of boundary problems in the theory of ordinary differential equations before the present century. The paper which he read at the International Congress in 1912 outlined some of the lines of advance during the intervening years. As Harvard exchange professor at

*Cf. his lecture before the International Congress at Cambridge.

Paris, in the winter term of 1913–14, it was especially fitting that he lay stress on the notions and methods of Sturm, who in the same city more than three quarters of a century earlier had in a brilliant memoir laid the foundation of this fundamental theory. Soon after his student days Bôcher had conceived the idea of writing a treatise on boundary problems as an elaboration of his encyclopedia article and had entered into a contract with Teubner for its publication. A considerable portion of this manuscript in German was left among his papers, but the contract had been cancelled a few years before his untimely death. The present volume embodies much of the material he had prepared for the other project.

The contributions of Americans to this field added to the felicity of the choice of the topic for the Paris lectures. In addition to the many contributions of the lecturer and his students, several other Americans, who a decade or more ago came under the influence of Hilbert, have done investigation in this field. Besides the notable advances made in the several papers by Birkhoff, other names such as Mason, Westfall, Kellogg, and Hurwitz might be mentioned in this connection. It is worthy of note also that a considerable proportion of the articles* dealing with this topic and published

* Among the papers touching on this field published since Bôcher's survey in 1912 and not referred to in his Leçons are the following: J. Tamarkine, "Sur quelques points de la théorie des équations différentielles linéaires ordinaires et sur la généralization de la séries de Fourier," *Rend. Cir. Mat. di Pal.*, vol. 34 (1912), pp. 345–382. Also see vol. 37 (1914), pp. 376–378 of same journal. G. D. Birkhoff, "Note on the expansion problems of ordinary linear differential equations," *Rend. Cir. Mat. di Pal.*, vol. 36 (1913), pp. 115–126. G. Hamel, "Ueber die lineare Differentialgleichung zweiter Ordnung mit periodischen Koeffizienten," *Math. Ann.*, vol. 73 (1913), pp. 371–412; also "Ueber das infinitäre Verhalten der Integrale einer linearen Differentialgleichung zweiter Ordnung, wenn die characteristische Gleichung zwei gleiche Wurzeln hat," *Mathematische Zeitschrift*, vol. 1 (1918), pp. 220–228. J. Yoshikawa, "Miszellen aus dem Gebiete der Oszillationsaufgaben," *Memoirs of the College of Science and Engineering, Kyoto Imperial University*, vol. 5 (1913), pp. 97–115. O. Haupt, "Ueber eine Methode zum Beweise von Oszillationstheoremen," *Math. Ann.*, vol. 76 (1914), pp. 67–104. L. Lichtenstein, "Zur Analysis der unendlichvielen Variabeln. Entwicklungssätze der Theorie gewöhnlicher linearer Differentialgleichungen zweiter Ordnung," *Rend. Cir. Mat. di Pal.*, vol. 38 (1914), pp. 113–166. D. Jackson, "Algebraic properties of self-adjoint systems," *Trans. Amer. Math. Soc.*, vol. 17 (1916), pp. 418–424. T. Fort, "Linear Difference and Differential Equations," *Amer. Jour. of Math.*, vol. 39 (1917), pp. 1–26; also "Some theorems of comparison and oscillation," *Bull. Amer. Math. Soc.*, vol. 24 (1918), pp. 330–334. R. D. Carmichael, "Comparison theorems for homogeneous linear differential equations of general order," *Annals of Math.*, vol. 19 (1918), pp. 159–171. H. J. Ettlinger, "Existence theorems for the general real self-adjoint

since these lectures were prepared, are from the pens of Americans.

The relation of this volume to the rest of Bôcher's work has already been discussed in this BULLETIN* by Birkhoff in his general survey and estimate of the author's contributions to mathematics. The particular task which he has set himself in the book under review is that of expounding the theory of the linear differential equation, keeping in mind not only the spirit of Sturm's work but, so far as possible, the letter also. Choosing to neglect some of the more direct and powerful but less elegant methods, he has given an admirable exposition of the analogy of the problems of linear differential equations to those of the purely algebraic linear equations.

Ever since Sturm was inspired by these analogies to undertake his investigations, an advance in the linear problem of algebra or of analysis has generally suggested a corresponding one in the other field. Bôcher's contributions to the literature of both the algebraic and analytic problems made it especially fitting that he undertake this presentation. It should, however, be pointed out that it is purely the parallelism between the two theories which he discusses. No use is made of either theory in the actual development of the other. There is, for example, no hint of the possibility of obtaining results by passing from the algebraic to the transcendental on letting the number of difference equations approximating to the differential equation increase without limit.

Material from the published memoirs of Bôcher plays no inconsiderable part in the subject matter of this volume. And as Birkhoff has already pointed out, there are two important

linear system of the second order," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 79–96. W. B. Fite, "Concerning the zeros of the solutions of certain differential equations," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 341–352; also "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivative," *Annals of Math.*, vol. 18 (1917), pp. 214–220. O. D. Kellogg, "Interpolation properties of orthogonal sets of solutions of differential equations, *Amer. Jour. of Math.*, vol. 40 (1918), pp. 225–234. R. G. D. Richardson, "Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order," *Amer. Jour. of Math.*, vol. 40 (1918), pp. 283–316. C. E. Wilder, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points," *Trans. Amer. Math. Soc.*, vol. 19 (1918), pp. 157–166; also see vol. 18 (1917), pp. 415–422 of same journal.

See also abstracts of two papers read before the meeting of the Amer. Math. Society in September 1919 by C. N. Reynolds, *Bull. Amer. Math. Soc.*, vol. 26 (1919), no. 2.

*February, 1919.

theorems which appear here for the first time. Below we shall discuss these theorems, one of which is concerned with an extension of the method of successive approximation and the other with the functional dependence of the solution on the coefficients of the equation and of the boundary conditions. A careful reading of these lectures gives the distinct impression also that a further filling in of gaps and a general rounding out of the theory is one of their admirable features.

The reader cannot fail to obtain from this volume an excellent insight into one of the fundamental bases of mathematical theory. It has the virtues and the faults of the lecture form, being interesting and suggestive but neither encyclopedic nor adapted for reference. It suffers from the fact that it was compiled by another; one notes the simplicity and elegance of ideas which were characteristic of Bôcher, but misses at times his lucidity of exposition. Sometimes the exact conditions under which a theorem is being developed are not easy to locate (pages 40, 71, 110) and it is occasionally difficult to find the formulation of results (pages 41, 42, 62, 81). The arrangement of material is not always the happiest and the passage from one topic to another is sometimes more abrupt than the division into sections would indicate. It may be that the war is in large measure responsible for a standard of proof-reading and typography somewhat lower than the very excellent one hitherto attained in the Borel series.

Before attempting to outline the argument of the text it may be well to consider briefly some of the properties of solutions of the special differential equation.

$$u'' + \lambda u \equiv d^2u/dx^2 + \lambda u = 0 \tag{1}$$

and its approximating difference equations. For this special problem we shall not only, as Bôcher would do, set up the parallelism of the two theories but we shall go beyond him in exhibiting the analogy of the actual mechanism and in indicating how by passing to the limit one theory goes into the other. The results are fairly typical of the general homogeneous equation of the second order and will serve as an illuminating introduction to the general theory.

There are two linearly independent solutions of (1), viz.,

sin $\sqrt{\lambda}x$ and cos $\sqrt{\lambda}x$, depending on the parameter λ and giving as the most general solution $u = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x$. The zeros of any two linearly independent solutions separate each other. If now one boundary condition is imposed, there can be only one linearly independent solution; for example, when $u(0) = 0$, the solution becomes $u = c_1 \sin \sqrt{\lambda}x$, having zeros at $\pm n\pi/\sqrt{\lambda}$. With increase of λ these zeros, or those of any solution, move closer together. It is readily seen that a second boundary condition can be imposed only in exceptional cases; the condition $u(1) = 0$ can, for example, be satisfied only if $\lambda = \pi^2, (2\pi)^2, \cdots$. The oscillation theorem for equation (1) under the boundary conditions

$$u(0) = u(1) = 0 \tag{2}$$

may be stated as follows: There exists one and only one characteristic number λ for which there is a non-identically vanishing characteristic solution of (1), (2) oscillating n times, that is, having n zeros in the interval. The infinity of positive characteristic numbers for $n = 1, 2, 3, \cdots$ diverges to infinity.

By dividing the interval (0, 1) into n equal parts and denoting by u_i the value at $x = i/n$, we can set up the algebraic equations approximating to (1) or (1) and (2). In the latter case they are

$$n^2(u_{i-1} - 2u_i + u_{i+1}) = -\lambda u_i, \qquad u_0 = 0, u_n = 0 \qquad (i = 1, \cdots, n-1). \tag{3}$$

Essentially we have here $n - 1$ homogeneous equations in $n - 1$ unknowns which will have a solution only when λ is one of the $n - 1$ roots of a determinant formed from the coefficients of the u's. The λ-matrix of the coefficients which is the type met with in studying pairs of bilinear or quadratic forms is here symmetric. We can call the roots characteristic numbers and state for the corresponding solutions a theorem of oscillation analogous to that for the differential equation.

If, in the approximation equations (3), n is given the value 5 and the u's on the left considered to be a different set of variables from the u's on the right, and the former solved in terms of the latter, these equations take the form

$$u_1 = \frac{\lambda}{25}(u_0 + \tfrac{4}{5}u_1 + \tfrac{3}{5}u_2 + \tfrac{2}{5}u_3 + \tfrac{1}{5}u_4 + 0u_5),$$

$$u_2 = \frac{\lambda}{25}(0u_0 + \tfrac{3}{5}u_1 + \tfrac{6}{5}u_2 + \tfrac{4}{5}u_3 + \tfrac{2}{5}u_4 + 0u_5),$$

$$(4) \qquad u_3 = \frac{\lambda}{25}(0u_0 + \tfrac{2}{5}u_1 + \tfrac{4}{5}u_2 + \tfrac{6}{5}u_3 + \tfrac{3}{5}u_4 + 0u_5),$$

$$u_4 = \frac{\lambda}{25}(0u_0 + \tfrac{1}{5}u_1 + \tfrac{2}{5}u_2 + \tfrac{3}{5}u_3 + \tfrac{4}{5}u_4 + u_5),$$

$$u_0 = 0, \quad u_5 = 0.$$

On denoting the range i/n $(i = 0, 1, 2, 3, 4, 5)$ by x_i and also by ξ_i, the matrix of coefficients on the right may be written

$$(5) \qquad K(x_i, \xi_j) = (1 - \xi_j)x_i, \text{ for } x_i \leqq \xi_j;$$

$$K(x_i, \xi_j) = (1 - x_i)\,\xi_j, \text{ for } x_i \geqq \xi_j,$$

and the solution (4) has the form

$$(4') \qquad u_j = \frac{\lambda}{5}\sum_{i=1}^{5} K(x_i, \xi_j)u_i \qquad (j = 0, \cdots, 5).$$

It will be noted that $K(x_i, \xi_j)$, which may be called the Green's function, is a symmetric matrix $[K(x_i, \xi_j) = K(\xi_j, x_i)]$, and further that the first difference quotient $5[K(x_{i+1}, \xi_j) - K(x_i, \xi_j)]$ for each j is one constant for $i \leqq j$ and another constant for $i > j$ and that the difference of these constants is in all cases unity. In other words, for $x_i = \xi_j$ the first difference of the matrix has a discontinuity equal to unity.

We may also set up a Green's function $K(x, \xi)$ for the differential equation (1) having the following properties: $K(x, \xi)$

(*a*) is symmetric in the variables (x, ξ),

(*b*) is continuous together with its first derivative, except that the latter has a break of 1 for $x = \xi$, and

(*c*) satisfies the relation

$$(6) \qquad u(x) = \lambda \int_0^1 K(x, \xi)\, u(\xi)\, d\xi.$$

This integral equation (6) is equivalent to the differential system (1), (2) and bears to it the same relation as equations

(4) and (4′) do to equations (3). The Green's function

$$(7)\qquad \begin{aligned} K(x, \xi) &= (1 - \xi)x \text{ for } x \leqq \xi, \\ K(x, \xi) &= (1 - x)\xi \text{ for } x \geqq \xi \end{aligned}$$

has essentially the same form (5) as that for the approximating algebraic equations (3).

The λ-matrix of any set of approximating equations such as (3) is equivalent to that of the corresponding set such as (4), both sets being taken as homogeneous. For any value of n, each of the characteristic numbers of (3) is equal to the corresponding one of (4). When n increases, each of the former approaches one of the infinite set of characteristic numbers $(n\pi)^2$ of (1), while each of the latter approaches one of the same infinite set of characteristic numbers of (6). The λ-determinant of (4) has for limit the symmetric Fredholm λ-determinant of (6). Thus by a passage to the limit $n = \infty$ the algebraic problem goes over completely into the transcendental.

After this consideration of a special problem, we are in a position to sketch the methods and results of the Leçons. Chapter II concerns itself with the parallelism between the linear differential and the linear algebraic systems. To a square array $|a_{ij}|$ there corresponds a set of linear homogeneous equations

$$(8)\qquad a_{i1}\xi_1 + a_{i2}\xi_2 + \cdots + a_{in}\xi_n = 0 \quad (i = 1, 2, \cdots, n)$$

and also another closely related adjoint set

$$(9)\qquad a_{1j}\eta_1 + a_{2j}\eta_2 + \cdots + a_{nj}\eta_n = 0 \quad (j = 1, 2, \cdots, n).$$

The rank of the determinant $|a_{ij}|$ being p, there are $n - p$ linearly independent sets of solutions of each set of equations. If $p = n$, there is only the identically zero solution and the systems are each incompatible. In this case the corresponding sets of non-homogeneous equations

$$(8')\quad \sum_{j=1}^{n} a_{ij}\xi_j = \gamma_i, \qquad (9')\quad \sum_{i=1}^{n} a_{ij}\eta_i = \delta_j$$

have each a unique solution. If $n - p = m \neq 0$ and the homogeneous sets (8) and (9) both have solutions, one must impose m linear conditions on the γ's (or δ's) in order that

the non-homogeneous system (8′) (or 9′) have at the same time a solution. The general solution of (8′) is the sum of a particular solution and of the most general solution of (8), the latter depending on $n - p$ constants.

Turning now to a discussion of the most general linear differential equation of the second order,

$$(10) \quad L(u) \equiv u'' + Pu' + Qu \equiv \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = R,$$

where P, Q, and R are continuous functions in an interval A, B, we observe that if no boundary conditions are imposed, the solutions can be approximated by $n - 1$ non-homogeneous difference equations which take the form

$$(11) \quad A_iu_{i-1} + B_iu_i + C_iu_{i+1} = D_i \quad (i = 1, \cdots, n - 1),$$

in which the $n + 1$ variables u are the values of the function at equally spaced points in the interval. Since there are two more variables than equations, the values of two of them may generally be assigned at random; in other words, the solution depends linearly on two arbitrary constants. In order to arrive at unique results, it would be necessary to impose two more linear conditions. As examples of such conditions the following may be cited:

(*a*) The values of the function or its derivative at two points a, b, of the interval may be pre-assigned; or more generally two linear conditions of the form

$$(12) \quad U_i(u) \equiv \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \beta_{i1}u(b) + \beta_{i2}u'(b) = \delta_i \quad (i = 1, 2),$$

may be imposed. More generally still these conditions may concern themselves with the values of u and u' at any number of points in the interval.

(*b*) Given the functions f_{i1}, f_{i2}, the solutions may be subject to the integral conditions

$$u_i(u) = \int_A^B [f_{i1}u(x) + f_{i2}u'(x)]dx = 0.$$

(*c*) There may be a combination of the types (*a*) and (*b*).

The equations (11) with two linear conditions such as (12) correspond to one of the sets (8′), (9′).

To the homogeneous equations (8) corresponds the homogeneous differential equation

$$(13) \qquad L(u) \equiv \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0,$$

together with two linear homogeneous equations of condition

$$(14) \quad U_i(u) \equiv \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \beta_{i1}u(b) + \beta_{i2}u'(b) = 0 \quad (i = 1, 2).$$

The differential equation adjoint to (10) is

$$(15) \qquad M(v) \equiv \frac{d^2v}{dx^2} - \frac{d(Pv)}{dx} + Qv = R$$

and to (13) is

$$(16) \qquad M(v) \equiv \frac{d^2v}{dx^2} - \frac{d(Pv)}{dx} + Qv = 0.$$

Similarly we may have boundary conditions adjoint to (12) or (14). In the latter case they are

$$(17) \qquad V_i(v) = 0 \qquad (i = 1, 2),$$

and we have the formula

$$(18) \qquad vL(u) - uM(v) = \frac{d}{dx}(Puv - Pv' + u'v).$$

The two systems (13), (14) and (16), (17) have the same number of linearly independent solutions or are both incompatible. The approximation equations of the two will bear to one another the same relation as (8) and its adjoint set (9), and the formula (18) has its analogue in a bilinear form involving the ξ's and η's. Precisely as in the case of the algebraic difference equations, the incompatibility of (13), (14) entails a unique solution of (10), (12) and of (13), (12) and of (10), (14). In case the system (13), (14) is compatible, a necessary and sufficient condition that there be a solution of the non-homogeneous system (10), (12) also is that certain linear equations of condition be imposed on R and the δ's, while the general solution is the sum of a particular solution and of the most general solution of (13), (14).

Every differential equation of the second order may, by

a change of variable, be put into a form which is self-adjoint, that is, such that (10) and (15) and also (13) and (16) coincide. The type most important in application is that where the boundary conditions are also self-adjoint; (14) and (17) would in that case be identical. This corresponds in the algebraic cases (8), (9) to a symmetric array $|a_{ij}|$.

The discussion of the simplest cases of the real solutions of the self-adjoint equation together with the behaviors of their zeros forms the topic of Chapter III. The properties of solutions of the most general homogeneous equation of this type, which can be taken in the form

$$(pu')' + gu = 0, \tag{19}$$

follow closely those concerning the solutions of (1) which were written in explicit form. The well-known theorem of Sturm concerning the alternation of the zeros of two linearly independent solutions u_1, u_2 of (19) follows from the readily established formula

$$\frac{d(u_1/u_2)}{dx} = \frac{u_2u_1' - u_1u_2'}{u_2^2} = \frac{\text{constant}}{pu_2^2}.$$

Conditions under which this separation theorem holds for two linear combinations of u and u' such as

$$\psi_1 = \varphi_1(x)u + \varphi_2(x)u', \quad \psi_2 = \varphi_3(x)u + \varphi_4(x)u'$$

are readily obtained. Of course, corresponding theorems for the approximating algebraic equations can also be deduced; but they are not of special interest.

The solutions of (19) being continuous functionals of p and g, we should expect that by proper choice of variation in these coefficients the zeros of $u(x)$ would move closer together as in the special case of equation (1), and this is a result easily established. Denoting by u_1 the solution corresponding to coefficients p_1 and g_1 in (19) and by u_2 that corresponding to p_2 and g_2, it follows from the identity, due to Picone,

$$\frac{d}{dx}\left[\frac{u_1}{u_2}(p_1u_2u_1' - p_2u_1u_2')\right]$$
$$= (g_2 - g_1)u_1^2 + (p_1 - p_2)u_1'^2 + p_2\left(u_1' - u_2'\frac{u_1}{u_2}\right)^2,$$

that, in an interval between two consecutive zeros x_1 and x_2 of u_1, a necessary condition that there exist a solution $u_2 \neq 0$ is that at least one of the expressions $p_1 - p_2$, $g_2 - g_1$ be positive. For otherwise on integrating the identity between x_1 and x_2, the two sides give opposite signs. Roughly speaking, it is true that if p decreases or g increases the zeros move closer together. An analogy can be found for the corresponding approximating difference equations.

By means of this comparison theorem the author easily finds upper and lower bounds for the lengths of intervals of oscillation by solving an equation with constant coefficients greater or less than p and g. For example, if $\gamma_1 = \min p > 0$, $\gamma_2 = \max g$, on solving

$$u'' + \frac{\gamma_2}{\gamma_1} u = 0,$$

it is found that if $\gamma_2 < 0$ the exponential result precludes the possibility of an oscillating solution of (19), while if $\gamma_2 > 0$, $\sin \sqrt{(\gamma_2/\gamma_1)}\, x$ gives a lower bound $\pi/\sqrt{\gamma_2/\gamma_1}$ for the interval between the zeros of that equation.

The most important special case which arises in applications is when $g = q + \lambda k$; that is, when (19) takes the form

$$(pu')' + (q + \lambda k)u = 0. \tag{20}$$

If p, q and k remain fixed and the parameter λ varies, the theorem of comparison just enunciated can be applied directly in case $k \geqq 0$ or $k \leqq 0$. In case $q \leqq 0$ and k changes sign it may be applied by dividing the equation through by $|\lambda|$. The remaining case where no restrictions are put on q and k involves complex solutions and has been worked out in detail only since Bôcher's lectures were given.*

Let us now impose one condition on the solution $u(x)$ of (20); for example,

$$c_1 u(a) + c_2 u'(a) = 0. \tag{21}$$

(This may, by a change of variable, be reduced to $u(a) = 0$ or $u'(a) = 0$.) By proper variations of p, q and λk the lengths of the intervals may be decreased and the zeros of u and u' made to move in toward $x = a$. In particular, when p, q and k are held fixed, with increase of $|\lambda|$ the number of

* Cf. Haupt and Richardson cited above.

zeros of $u(x)$ in the interval a, b can be made to increase without limit. Further there is a set of values of $\lambda = \lambda_1, \lambda_2, \cdots$ (characteristic numbers), which diverge to infinity, to which correspond the characteristic solutions $u(x)$ satisfying (21) and a second linear condition

$$c_3u(b) + c_4u'(b) = 0. \tag{21'}$$

In case $k \geqq 0$ the oscillation theorem for the system (20), (21), (21') will be approximately that for the special system (1), (2); but in case k has both signs the result must be so modified as to prescribe two characteristic numbers (one positive and the other negative) corresponding to which are solutions having any given number of zeros in the interval.

Besides a discussion of conditions under which the characteristic numbers of a differential equation will be real and the derivation of an oscillation theorem for a simple case of n differential equations each with n parameters, Chapter IV contains a detailed treatment of the oscillation theorem for the interesting periodic conditions

$$u(a) = u(b), \qquad u'(a) = u'(b). \tag{22}$$

It may be remarked that when more general boundary conditions of the type (14) are imposed, it is customary to follow Hilbert in reducing them to a small number of normal forms and in discussing these in sequence. It may also be noted that the theorem of oscillation derived by Bôcher holds more generally for all boundary conditions belonging to the same normal form as (22). For $k \geqq 0$ the theorem is precisely that which one would surmise from a slight consideration of the special equation (1) under these same conditions (22), viz.: corresponding to each even number of oscillations there are two characteristic solutions, to each odd number there are none, and to zero there is one.

Chapter V begins with a consideration of the existence and fundamental properties of the Green's function for differential equations of the nth order. Since the case $n = 2$ is typical, we may here confine ourselves to the equations already discussed. When the homogeneous system (13), (14) is incompatible, a Green's function $G(x, \xi)$ having the properties (a), (b), (c) enumerated above may be set up and it furnishes as a unique solution of (10), (14) the formula

$$u(x) = \int_a^b R(x)G(x, \xi)d\xi. \tag{23}$$

The Green's function for the adjoint system (15), (17) turns out to be $G(\xi, x)$, so that for a self-adjoint system it is symmetric in x and ξ, as we saw in the special form (7) set up for the system (1), (2). When a solution of the system (10), (12) is desired, the formula (23) must be modified by the addition of certain auxiliary terms. While no mention is made of the fact in the lectures, these theorems have an analogue in the theory of algebraic equations.

The same sort of argument as establishes (23) leads to a proof of the equivalence of the non-homogeneous differential system (10), (12) with the non-homogeneous integral equation

$$u(x) = f(x) + \int_a^b R(x)G(x, \xi)u(\xi)d\xi,$$

where $f(x)$ is the sum of certain auxiliary terms. In case the differential system is homogeneous, the term $f(x)$ is zero and (13), (14) is equivalent to the corresponding homogeneous integral equation

$$u(x) = \int_a^b R(x)G(x, \xi)u(\xi)d\xi$$

of which (6) is a special case.

In Chapter I the author proves by the method of successive approximations the usual existence theorem for the solution of the non-homogeneous equation (10) under the one-point boundary condition

$$u(c) = \gamma, \qquad u'(c) = \gamma' \tag{24}$$

by showing that the series

$$v_1 + v_2 + v_3 + \cdots; \qquad v_1' + v_2' + v_3' + \cdots$$

approximating to u and u' are uniformly convergent. This is done by establishing the inequality

$$\left.\begin{matrix} |v_n| \\ |v_n'| \end{matrix}\right\} \leqq \frac{CL^{n-2}(2M)^{n-2}(x-c)^{n-2}}{(n-2)!},$$

where C is greater than $|v_1|, |v_1'|, |v_2|, |v_2'|$; M greater than $|P|, |Q|, |R|, |\gamma|, |\gamma'|$; and L greater than both unity and the length of the interval.

One of the contributions of Bôcher first appearing in these lectures is the answer to the question as to the nature of the

dependence of u on the coefficients P, Q, R, and the constants γ, γ'. He shows that the solution u is a continuous *functional* $F(P, Q, R, \gamma, \gamma')$, and moreover that this is true even when the three arguments P, Q, R have a finite number of discontinuities. It may be assumed that as in the case of functions the sum or product of two continuous functionals is a continuous functional as is also the integral of a functional. The reasoning of Bôcher, which is characteristically simple and elegant, then proceeds as follows. Since each v and v' is obtained by a finite number of integrations of explicit sums and products of P, Q, R, γ, γ', and of specified continuous functions of x, it is a continuous functional of the arguments. The finite sums $\sum_1^{n_1-1} v_n$, $\sum_1^{n_1-1} v_n'$ will then also be continuous functionals for any definite index n_1. But for the functional field limited by the assumed value of the constant M, and for an index N_1 large enough, $\left|\sum_{n_1}^{\infty} v_n\right|$ and $\left|\sum_{n_1}^{\infty} v_n'\right|$ will be less than any preassigned $\epsilon > 0$ for $n_1 > N_1$. This establishes the theorem.

Returning to the topic of successive approximations at the end of Chapter V, Bôcher by an application of the Green's function fulfils the promise made in his International Congress lecture of considering a generalization of the theory of successive approximations. Taking the differential system in the form

$$(25) \qquad L'(u) = L''(u) + R, \qquad U_i'(u) = U_i''(u) + \delta_i,$$

where L' and L'' are homogeneous linear differential expressions of orders n and $m < n$ respectively and $U_i'(u)$ and $U_i''(u)$ are linear forms in $u(a), u'(a), \cdots, u^{(n-1)}(a), u(b), u'(b), \cdots, u^{(n-1)}(b)$ and assuming that the homogeneous system $L'(u) = 0, U_i'(u) = 0$ is incompatible, he finds that the series set up by the usual method of successive approximations as solutions of the successive equations

$$L'(u_m) = L''(u_{m-1}) + R, \qquad U_i'(u_m) = U_i''(u_{m-1}) + \delta_i$$
$$(m = 1, 2, \cdots)$$

may or may not converge uniformly. The discussion of this question of convergence is made to depend on the charac-

teristic values of the system

$$(26)\qquad \begin{aligned} L'(u) &= \lambda[L''(u) + R''] + R', \\ U_i'(u) &= \lambda[U_i''(u) + \delta_i''] + \delta_i', \end{aligned}$$

where $R'' + R' = R$ and $\delta_i'' + \delta_i' = \delta_i$, and (26) reduces to (25) for $\lambda = 1$. If for u_0 we take the solution of $L''(u_0) + R'' = 0$, $U_i'' + \delta_i'' = 0$, the method of successive approximation gives us a power series in λ which will converge within a certain circle (finite or infinite) with $\lambda = 0$ as center and which will diverge without. To obtain a knowledge of the radius of this circle, Bôcher considers a differential system

$$(27)\qquad L(u) = R, \qquad U_i(u) = \gamma_i \quad (i = 1, 2, \cdots, n),$$

of which (26) is a special case. The coefficients are assumed continuous in the real variable and analytic in λ for a certain Weierstrass domain. His profound grasp of the algebraic problem leads him to a consideration of the function

$$(28)\qquad \frac{\begin{vmatrix} u_0 & y_1 & y_2 & \cdots & y_n \\ U_1(u_0) - \gamma_1 & U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ U_n(u_0) - \gamma_n & U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix}}{\begin{vmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdot & \cdots & \cdot \\ U_n(y_1) & \cdots & U_n(y_n) \end{vmatrix}},$$

where u_1 is a solution of $L(u) = R$ and $y_1, \cdots, y_n$ are a fundamental system of solutions for the homogeneous equation $L(u) = 0$. When λ is not a characteristic number of the homogeneous system $L(u) = 0$, $U_i(u) = 0$, in other words when the denominator is not zero, a substitution shows that (28) is a solution of (27). This solution is continuous in x and analytic in λ except for the characteristic numbers.

But Bôcher is able to proceed further. Even when the denominator of (28) vanishes, as it will for the characteristic number λ_i, it may happen that the numerator has a zero of the same order, in which case a solution of the system (27) still exists for this value of λ. For such a characteristic number both the homogeneous and non-homogeneous systems must have solutions. He had already noted above the corresponding condition of affairs in the algebraic problem, where

the solutions of the homogeneous equations (8) are also solutions of the adjoint non-homogeneous equations (9′). This algebraic condition which may be expressed by the formula $\Sigma\delta_j\xi_j = 0$, where ξ_j denotes any solution of (8), has its analogues in the transcendental case.

He is now ready to state the results for (26) and hence for (25). If the circle of convergence of the power series in λ is not infinite, it passes through one of the characteristic values of the homogeneous system

$$L'(u) = \lambda L''(u), \qquad U_i'(u) = \lambda U_i''(u) \tag{29}$$

corresponding to (26). In the ordinary case this characteristic number will be that of smallest absolute value; but if it is not, each solution of (29) corresponding to a λ of smaller absolute value must have index and multiplicity equal, that is, both the homogeneous and non-homogeneous equations have solutions; and at least one solution corresponding to a point on the circle must have them unequal. If the circle of convergence has a radius greater than unity, the approximating series gives a solution of (25). For one-point boundary conditions, such as (24), there are no characteristic numbers and the series converges everywhere.

It may be remarked that the ordinary case of the theorem just given might be conjectured from a consideration of the problem from the standpoint of the integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u(\xi) d\xi \tag{30}$$

equivalent to (26). By successive substitution of this equation into itself we get the power series

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi) f(\xi) d\xi + \lambda^2 \int_a^b K(x, \xi) \int_a^b K(\xi, \xi_1) f(\xi_1) d\xi_1 d\xi + \cdots.$$

This is the well-known Neumann-Liouville series solution of the integral equation (30), provided it converges uniformly, as it certainly will inside of any circle extending out to the nearest characteristic number of the corresponding homogeneous integral equation. This power series in λ must be identical with that obtained from (26) by successive approximations.

R. G. D. RICHARDSON

DICKSON'S HISTORY OF THE THEORY OF NUMBERS.

History of the Theory of Numbers. By LEONARD EUGENE DICKSON. Volume I. Carnegie Institution of Washington, 1919. 486 pp.

IN these days when "pure" science is looked upon with impatience, or at best with good-humored indulgence, the appearance of such a book as this will be greeted with joy by those of us who still believe in mathematics for mathematics' sake, as we do in art for art's sake or for music for the sake of music. For those who can see no use or importance in any studies in mathematics which do not smack of the machine shop or of the artillery field it may be worth while to glance through the index of authors in this volume and note the frequent appearance of names of men whose work in the "practical applications" of mathematics would almost qualify them for a place on the faculty of our most advanced educational institutions. One of the most assiduous students of the theory of numbers was Euler, whose work otherwise was of sufficient importance to attract the notice of a king of Prussia. The devotee of this peculiar branch of science can reassure himself that he is in very good company when he reads the list of authors cited in connection with the famous theorems of Fermat and Wilson in Chapter III; Cauchy, Cayley, d'Alembert, Dedekind, Euler, Gauss, Jacobi, Kronecker, Lagrange, Laplace, Legendre, Leibniz, Steiner, Sylvester, Von Staudt and a host of others, great and small, living and dead, to the number of over two hundred, who have found these absolutely "useless" theorems worthy of their most serious attention. The reviewer is firm in the faith that no great headway will ever be made in any science, least of all in mathematics, by those who are always looking for the penny. He takes comfort also in the fact that great teachers are not found among those who are scornful of mathematics for mathematics' sake. Their race is not likely to be perpetuated, and the chances are that the study of the theory of numbers will become increasingly popular as the years go by.

One is struck in glancing through the book by the remarkable combination of superstition, fancy, scientific curiosity, and patient, plodding experiment that has figured in advancing the science of the theory of numbers. Thus, in the first chapter, which has to do with the theory of perfect numbers, the first name that appears is that of Euclid, who proved that if $p = 1 + 2 + 2^2 + \cdots + 2^n$ is a prime then $2^n p$ is a perfect number, that is, a number which is equal to the sum of its aliquot parts. This solid contribution of Euclid's is followed by fanciful speculations regarding the ethical import of such numbers! "Alcuin of York and Tours explained the occurrence of the number 6 in the creation of the universe on the ground that 6 is a perfect number. The second origin of the human race arose from the deficient number 8. Indeed, in Noah's ark there were eight souls from which sprung the entire human race, showing that the second origin was more imperfect than the first, which was made according to the number 6."

The requirement that p be a prime was soon overlooked by such mystics and we have a long series of writers who state that perfect numbers always end alternately in 6 and 8, and that between any two successive powers of 10 one such number is always to be found. These errors, which would have been discovered by a little patient experiment, persist among writers on the subject till the days of experimental workers like Cataldi, who noted that the fifth and sixth perfect numbers both end in 6. To get this result it was necessary for him to show that 8191 and 131071 were both primes, which he did by the straightforward method of trying as divisors every prime less than their respective square roots. Thus early in the theory arises the fundamental problem; to discover the prime factors of a given number. It was the discovery of a method for factoring such numbers as $a^n \pm 1$ that gave Fermat such power in the investigations concerning perfect numbers, so that, for example, he was able to state that there is no perfect number of 20 or of 21 digits, contrary to the opinion of those who believed that there was a perfect number between any two powers of 10. The efforts of mathematicians since Fermat have resulted in identifying twelve perfect numbers of the type $2^{n-1}(2^n - 1)$ corresponding to the following twelve values of n: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127. The work still goes on. $2^{67} - 1$ was proved

composite by Lucas in 1876. The actual factors were not found, however, till 1903, when Professor Cole found them to be 193707721 and 761838257287. The existence of odd perfect numbers has not been as yet proved or disproved.

Chapter II gives the history of the formulas for the number and sum of the divisors of a number, together with the problems proposed by Fermat (*a*) to find a cube which when increased by the sum of its aliquot parts becomes a square, and (*b*) to find a square which when increased by its aliquot parts becomes a cube. A third problem, due to Wallace, is also treated: to find two squares, other that 16 and 25, such that if each is increased by the sum of its aliquot parts the resulting sums are equal. Among the contributors to the history of these matters we note Cardan, Mersenne, Newton, Waring, Descartes, Euler, Kronecker, and others.

Chapter III is devoted to Fermat's and Wilson's theorems, their converses and their generalizations, together with theorems on the symmetric functions of $1, 2, 3, \cdots, p-1$ modulo p. The chapter begins with the astonishing statement that the Chinese seem to have known as early as 500 B.C. that $2^n - 2$ is divisible by n (n a prime). This important theorem, rediscovered some two thousand years later by Fermat, has been the center of an immense amount of activity, beginning with Leibniz who left a proof of it in manuscript. Whether, as Mahnke, who made a careful study of the Leibniz manuscripts seems to believe, Leibniz discovered the theorem independently, before he became acquainted in 1681–2 with Fermat's Varia Opera of 1679, or whether he heard of the theorem when he was in Paris in 1672 or when he was in London in 1673 is a question worth study. There would seem to be no reason to doubt, however, that the manuscript proof of Leibniz is the earliest known, and that to Leibniz also belongs the discovery as early as 1682 of the theorem known as Wilson's theorem, the first published proof of which was given by Lagrange, nearly a century later.

The function $a^{n-1} - 1$ which Fermat found to be always divisible by n is sometimes divisible by n^2, as for example when $n = 11$ and $a = 3$. The question as to when this phenomenon appears was raised by Abel, answered with numerical examples by Jacobi, and studied by Eisenstein, Sylvester and many others. The question has a bearing on Fermat's last theorem, as is shown by Wieferich's theorem that if $x^n + y^n + z^n = 0$

is satisfied by integers x, y, z, prime to n (n an odd prime) then $2^{n-1} - 1$ is divisible by n^2. Chapter IV gives the literature on this subject.

Chapter V is another long one on Euler's φ-function and generalizations of it, together with the related theory of Farey series. Few other functions of analysis have been studied with such enthusiasm by mathematicians of the first rank, and the many remarkable applications of it to other branches of analysis and to geometry are well indicated in this chapter. Euler's function has had an amusing history as to its name and the notation for it. Euler himself gave it no name and at first no notation, later using the notation $\pi(n)$ to denote the number of integers less than n and prime to it. With this definition, of course, $\pi(1) = 0$. If we define the function as the number of integers not greater than n and having with n the greatest common divisor 1 then $\pi(1) = 1$ and this value of the function of unity fits most easily in with formulas connected with it. Gauss introduced the symbol $\varphi(n)$, which seems to have a good chance of becoming permanent. Prouhet proposed the name indicator and the notation $i(n)$, which has had some adherents among French writers in spite of the fact that the name was already preempted for another function by Cauchy. Sylvester named the function the totient function and denoted it by $\tau(n)$. This name with Gauss' notation has been very extensively used among American writers. The literature of this function is so great that students and teachers will find this chapter very valuable, as also the shorter Chapter VI on periodic decimal fractions. An immense amount of work is necessary to discover what others have done in these fields.

Among the by-products in the study of expressions of the form $a^n - 1$, which is itself a by-product of the theory of perfect numbers, are the theories of primitive roots, binomial congruences, more general congruences, Galois imaginaries, and periodic decimal fractions. These matters are treated in Chapters VI, VII, VIII. Chapter VI, on periodic decimal fractions, and Chapter XX on properties of the digits of numbers, as well as much of Chapter XI, come under what might perhaps be called the metrical theory of numbers, having to do with properties which depend on the base employed to represent them.

The theory of the divisibility of factorial expressions

seems to have been studied for the first time by Leibniz, who noted in his manuscript that the multinomial coefficients except the first appearing in the expansion of $(a + b + c + \cdots)^n$, where n is a prime, are divisible by n. Legendre followed with a formula for the highest power of a prime to be found in $m!$. The later contributions to this subject are found in Chapter IX.

One of the most interesting of Euler's many discoveries is the formula relating to $\sigma(n)$, the sum of the divisors of n. The values of this function for $n = 1, 2, 3, 4, 5, 6, 7$ are 1, 3, 4, 7, 6, 12, 8 and no one but an Euler would have been able to find any simple law connecting such an irregular series of numbers. It is interesting to find him with his eye always turned toward the fundamental problem of finding the factors of numbers. Even this formula he uses to prove that 101 is a prime. He finds that his formula gives him 102 for the sum, whence he concludes that 101 is a prime. This method should be, but is not, listed among the tests for primality in Chapter XVIII. As it stands it is of no practical use for that problem, but neither is Wilson's theorem, for that matter. In any event it is remarkable that the primality of a number should be made to depend on the sums of the divisors of a certain set of smaller numbers. Chapter X is particularly valuable as giving a list of formulas, scattered through many journals, some of which are given by one author without proof and proved or disproved by others. It is difficult for the ordinary worker to run them down. The theory of partitions, which has such important connection with this subject, is to be given a chapter in Volume II.

Chapter XI is a list of miscellaneous theorems on divisibility and theorems on the greatest common divisor and the least common multiple. Here is indeed a mixture of important work like the results of Cesàro, Gegenbauer, de la Vallée Poussin, Landau, Dedekind and Kronecker, side by side with the amusing note that the consecutive numbers, 242, 243, 244, 245 have each a square factor greater than unity. Here are found also many approximative or asymptotic formulas such as $6x/\pi^2$ for the number of integers not greater than x and divisible by no square greater than unity, the error being less than the square root of x. The history of the familiar rule for casting out 9's and 11's is reserved for Chapter XII, which after giving a faithful account of this

matter winds up with a list of over two pages of titles not reported on.

The history of factor tables and lists of primes given in Chapter XIII begins again in the remote days of Eratosthenes. The net result of over twenty centuries of labor seems to be that the list of primes up to ten million is determined with a high degree of accuracy, thanks to the work of men drawn from many different races and nations. Some day, perhaps, a machine may be constructed to extend the list further, but the methods used for computing the tables already in existence will hardly serve for higher limits. It should be clearly understood by anyone who contemplates further work in this field that the most troublesome and tedious part of the work lies not so much in the computation as in the actual printing and proof-reading of the results.

There is perhaps no problem in the theory of numbers more fascinating to the scientist, or wider in its appeal to all sorts and conditions of men, than the problem of finding a method for factoring numbers which shall be better than the straightforward one of trying as divisors the primes less than the square root. The invention of a new instrument for studying the heavens must appeal in much the same way to the astronomer. One can easily imagine the delight which Fermat must have taken in his method of factoring a number by expressing it as the difference of two squares. It is one of the few discoveries of the illustrious Frenchman which he condescends to describe in detail. He later improved on this method and found others much more powerful for dealing with numbers of certain forms. Euler was also an indefatigable worker in this direction. He showed by some six pages of calculations that the number 1000081 is expressible in only one way as the sum of two squares and so must be a prime. "Dolendum autem est" he mourns, "hanc methodum non ad omnes numeros explorandos adhiberi posse." He proceeded to extend the method to representation by means of other binary quadratic forms, thus developing the most powerful tool now in our possession for this purpose. Legendre supplemented this method by employing the continued fraction for finding representations of the number. Gauss invented a method of exclusion, and Seelhoff made tables for the same purpose. Many other methods have been proposed, some of which are very successful if the factors are related in particular ways.

An account of all these methods is given in Chapter XIV, while in Chapters XV and XVI appear methods applicable to numbers of special forms like $2^{2^n}+1$ and $a^n \pm b^n$.

In the early years of the thirteenth century Leonardo Pisano noticed the series of numbers 1, 2, 3, 5, 8, 13, $\cdots$, each of which is the sum of the two preceding it. The study of this series and of others like it has yielded many important results. Lucas has hit upon a general theory which includes these series. The whole theory is connected with the theory of linear difference equations with constant coefficients and also with the theory of continued fractions. Chapter XVII gives the history of this important subject.

The existence of an infinitude of primes has been known since Euclid. That there are an infinite number of them in any arithmetical progression $mx+n$ where m and n are relatively prime was not proved until 1837, when Dirichlet established it by a very difficult analysis. Dirichlet also found that any primitive binary quadratic form can represent an infinitude of primes. Much important work centers in these great discoveries. Of equal difficulty and importance is the problem of finding either exactly or by an approximative formula the number of primes between given limits. The history of these problems is contained in Chapter XVIII together with the literature connected with Goldbach's conjecture, still neither proved or disproved, that every even number is the sum of two primes. Bertrand's postulate, proved by Tchebycheff, that there is at least one prime between x and $2x-2$ for x greater than 3 is also treated in this chapter. One can not fail to be impressed with the immense fields of analysis drawn upon in the attacks on these problems.

Chapter XIX gives the history of the function $\mu(n)$ of Moebius which is useful in the inversion of series, and plays an important rôle in the derivation of many approximative formulas.

The last chapter (XX) is devoted to the listing of many curious and amusing properties of numbers, properties chiefly connected with the representation to the base 10. It is hardly likely that any important results will flow from the study of problems like finding numbers like $512=(5+1+2)^3$ but such little things serve sometimes to attract students to more serious things.

It may, perhaps, be objected that the book is not so much

a history as a list of references from which a history of the theory of numbers might be written. Be that as it may, there is the greatest need for just such a piece of work to promote efficiency among the professional workers in this field and to prevent them from wasting their time on problems that have already been adequately treated, and also to suggest other problems which still defy analysis. It is to be hoped that the second volume will not be long delayed.

D. N. LEHMER.

THE CALCULUS OF PROBABILITY.

Calcolo delle Probabilità. By GUIDO CASTELNUOVO. Albrighi, Segati & C., Milano-Roma-Napoli, 1919. xxiii + 373 pp.

THE increased interest in the calculus of probability which has arisen during the past fifty years has been due in no small part to the brilliant applications of it in the field of physical phenomena. One of the most important of these, first in point of time and a model for the others, is due to Maxwell and has to do with the distribution of velocities of the molecules of a gas. Some of these investigations of physical phenomena on the basis of the laws of probability, operating under an assumed absence of determining physical laws among certain groups of phenomena, have been so successful in accounting for or predicting physical events that the conception has arisen in some quarters of the "laws of nature" as merely certain statements of average among fortuitous occurrences. It is almost uncanny to find relatively constant results of measurements of certain sorts predicted by a mathematical analysis based essentially on an assumption of chance distribution; and yet this is found in not a few important investigations.

A paradoxical situation of this sort will always excite interest. The human mind is peculiarly uncomfortable in the presence of a demonstrated result and an intuitive feeling between which there seems to be disharmony. A disturbance of our equilibrium is produced when we see the theory of probability thus accounting for what seemed to be fixed relations among phenomena. Where there is lack of equilibrium there is

reckless thought; and in the presence of the latter new analyses are made of the established results of science relevant to the questions raised. This perhaps accounts in some measure for the large number of treatises on the theory of probability published in recent years.

It is interesting to observe that in the field where these physical applications were first made and where they have been most successful perhaps in relating physical phenomena by means of theoretical investigations there is yet a feeling of dissatisfaction with the situation from the critical point of view of rigorous thinking. A minute examination of the various demonstrations proposed for a justification of Maxwell's law of distribution of velocities brings out the fact that the reasoning, though meriting the attention which it has received and in its first aspects appearing persuasive if not conclusive, is nevertheless not immune to criticism. In fact there is sufficient doubt concerning the logical processes, in any of the several proofs advanced, to leave one with a desire for some additional support—so much so, that a frank and conscientious statement would probably bring out the confession that one is somewhat relieved by the fact of the success of the theory in accounting for physical phenomena.

This most noteworthy instance which arises in the applications may serve to emphasize the oft-mentioned fact that great logical difficulties are present in the whole theory of probability. It manifests itself from the outset, being present to start with in the definition of probability. The difficulty has so persisted here, in spite of all the careful analysis which has been made and the varying form of definition which has been proposed, that one cannot yet avoid a feeling of discomfort from the start. In a range of ideas having no contact with other things and no applications in other fields, such a feeling of discomfort would do much to inhibit a detailed development of theory; one would be disposed to labor longer to clear up the initial logical difficulties. But in this case there are so many points of contact with other interests and during the past fifty years such an increasing suggestiveness of fundamental value in physical investigations that one is disposed to lay aside temporarily his feeling of discomfort and to continue to proceed tentatively with the theory till he is able to check it up through a testing of the results.

When one proceeds thus in a range of ideas somewhat

lacking in clarity and in logical precision, it is peculiarly important that the significance of the initial ideas, the character of the difficulties, and the steps of the argument, shall be rendered as clear as possible through a suitable exposition. The author of the present treatise is particularly successful in making clear at the beginning of each section the problem which is to be treated in that section and then in summing up his results clearly at the proper place. Moreover the more difficult or more elusive notions are treated with care and the learner is not only presented with a correct statement but is also warned against certain probable errors. Illustrative cases are well chosen for bringing out clearly the character of the notions in consideration. If all exposition maintained in this respect the same felicity as this book, students of mathematics would be saved a great deal of energy not infrequently diverted from a mastery of the matter by the necessity of an undue effort to follow the author's form or order of presentation.

Several subjects of mathematics, depending on a minimum of technical information and detail though requiring maturity for their mastery (like the calculus of probability), would become accessible to a wider range of readers interested in science if presented in such way as to be more readily understood. Such persons, if they attempted to read the current expositions, would now too often be repelled by the forbidding style of a condensed exposition, frequent in technical mathematical works, which wastes much more energy and of greater value than that which would be expended in printing the somewhat larger volumes needed for an exposition more carefully serving the reader's comfort. It is poor economy when a little printer's ink and paper are saved at the expense of an increase in the expenditure of scientific energy needed to read an exposition which has been too much condensed through improper omissions.

Two means are used in the present work to keep the main body of the book accessible to readers with a minimum mathematical training: a considerable part of the more technical matter is put in the appendix of sixty-four pages; a few articles are marked with an asterisk and these may be omitted without destroying continuity. The latter sometimes give new results; at other times they put on a more rigorous or more exact basis what had already been developed in a more intuitive way and

with less care for completeness. An instance of the latter sort is afforded by sections 45 and 46, where one has a more satisfactory derivation and discussion of certain approximations earlier gotten in a somewhat formal way. The less mature reader may omit these more critical sections; he will perhaps not be dissatisfied with the earlier work, where approximate formulas are used rather too much as exact formulas are employed.

About half of the nearly three hundred pages in the main body of the work is devoted to the more elementary and more general aspects of the theory. Here the exposition is particularly marked by that clarity of expression which is characteristic of the entire work. The reader is led in a pleasing way through such topics as probability and frequency, total and composite probability, mathematical expectation and mean value, the problem of repeated trials and the theorem of Bernoulli, the normal law of probability, and probability involving continuous distribution. Then one meets in the second half of the text brief chapters on probability of causes, statistical interpretation, errors of observation, method of least squares, and the kinetic theory of gases. The greater portion of the appendix is given to a careful consideration of the theorem of Laplace-Tchebychef and an extension of it.

R. D. CARMICHAEL.

CORRECTIONS.

THE following errata have come to the attention of the editors of the BULLETIN:

In Professor Brouwer's paper: "Intuitionism and formalism," in volume 20, pages 81–96,

page 91, in the footnote, *for:* the proof *read:* the formalistic proof of this property; and interchange "können wir bestimmen" with "muss es geben;"

page 94, line 4, *for:* but that it is impossible *read:* and when a proof is added (for the rest not recognized by the intuitionist) that it is impossible.

In Professor Forsyth's paper: "Formulas for constructing abridged mortality tables for decennial ages," in the current volume,

page 35, formula (1), w_0 should be w_0/t;
page 37, formula ($5i'$), the last + sign should be −;
page 38, formula ($5t$), the coefficient $6\frac{1}{2}$ should be $5\frac{1}{2}$;
page 38, formula ($5t'$), the last + sign should be −.

In Professor Hayashi's paper: "On the rectifiability of a twisted cubic," in the current volume,
page 74, line 8, *for* $\dot{x}\dot{y}\dot{z}$ *read:* $|\dot{x}\dot{y}\dot{z}|$.

In Professor Carmichael's paper: "Note on a physical interpretation of Stieltjes integrals," in the current volume, pages 102–105, the last paragraph of the paper should be deleted.

NOTES.

THE twenty-sixth annual meeting of the American Mathematical Society will be held in New York City on Tuesday and Wednesday, December 30–31, 1919, immediately preceding the meeting of the Mathematical Association of America. The annual election of officers of the Society will close on Wednesday morning. The joint dinner of the two organizations will be held on Wednesday evening. Titles of papers for this meeting should be in the hands of the Secretary by December 9.

THE thirteenth western meeting of the Society, combining the meetings of the Chicago and the Southwestern Sections, will be held at St. Louis in affiliation with the American Association for the Advancement of Science on December 30–31. Retiring addresses will be delivered by Professor G. D. BIRKHOFF, as chairman of Section A, and Professor G. A. BLISS, as chairman of the Chicago Section. Titles and abstracts of papers should be in the hands of Professor ARNOLD DRESDEN, secretary of the Chicago Section, by December 2.

A NEW edition of the List of Officers and Members of the Society is in preparation and will be issued in January next. Blanks for furnishing information have been sent to the members, and prompt response is requested in order that the List may be as accurate as possible.

THE opening (September) number of volume 21 of the *Annals of Mathematics* contains the following papers: "Investigation of a class of fundamental inequalities in the

theory of analytic functions," by J. L. W. V. JENSEN, translated by T. H. GRONWALL; "Functions of limited variation in an infinite number of dimensions," by P. J. DANIELL; "A new sequence of integral tests for the convergence and divergence of infinite series," by R. W. BRINK; "Calculation of the complex zeros of the function $P(z)$ complementary to the incomplete gamma function," by P. FRANKLIN; "Total differentiability," by E. J. TOWNSEND.

THE following nine doctorates with mathematics as major subject were conferred by American universities in the academic year 1918–1919; the title of the dissertation is added in each case: LOUIS BRAND, Harvard, "I. On linear equations with an infinite number of variables. II. On infinite systems of linear integral equations. III. Flexual deflections and statically indeterminate beams"; G. H. CRESSE, Chicago, "On the class numbers of binary forms;" JESSIE M. JACOBS, Illinois, "The trilinear binary form as a cubic surface;" GERTRUDE I. MCCAIN, Indiana, "Series of linear iterated fractional functions—character of the functions;" C. N. REYNOLDS, Harvard, "On the zeros of solutions of linear differential equations;" WAYNE SENSENIG, Pennsylvania, "The invariant theory of involutions on conics;" CHAN-CHAN TSOO, Harvard, "The geometry of a non-euclidean line-sphere transformation;" C. L. E. WOLFE, California, "On the indeterminate cubic equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$;" H. E. WOLFE, Indiana, "A study of plane circle to circle transformations by means of tetracyclic coördinates."

AT the meeting of the National academy of sciences at New Haven, November 10–12, the papers read included the following: By E. W. BROWN: "Jupiter's five attendant planets;" by EDWARD KASNER: "Some new theorems in the dynamics of particles;" by J. K. WHITTEMORE: "The starting of a ship under constant power."

AT the University of Strasbourg, the following courses in pure and applied mathematics are announced for the academic year 1919–1920:—By Professor FRÉCHET: Functional calculus, two hours first semester, three hours second semester; Approximating functions, one hour first semester.—By Professor VALIRON: Entire functions, two hours second

semester.—By Professor VILLAT: Rational mechanics, three hours first semester, one hour second semester; Elliptic functions with application to mathematical physics, two hours second semester.—By Professor ESCLANGON: Astronomy, two hours. Courses in pure mathematics will also be given by Professor PÉRÈS. MM. ANTOINE and DARMOIS are maîtres de conférences in mathematics, and M. VÉRONNET is chargé de conférences in rational mechanics.

VUIBERT, of Paris, has announced that publication of the following periodicals is to be resumed this fall: *L'Education mathématique* (which last appeared in July, 1914) and *Revue de Mathématiques spéciales* (last published in September, 1914).

THE Belgian academy of sciences has awarded the Deruyts prize to Dr. LUCIEN GODEAUX, for his researches in algebraic geometry. The Academy states that an extension of time has been granted during which memoirs may be presented for two prizes, of 800 francs each, first announced March 7, 1914 for award August 1, 1915. The subjects are: The infinitesimal geometry of curved surfaces, and Systems of conics in space. Competing memoirs should be written in French, Flemish, or Latin, and should be sent to the Secretary at the Palais des Académies, Brussels, before August 1, 1920.

THE decennial prize in pure mathematics founded by the Belgian government has been awarded to Professor A. DEMOULIN, for the decade 1904–1913.

THE Paris academy of sciences announces the award of the following prizes in pure and applied mathematics: the Francœur prize (1,000 francs) to Dr. G. GIRAUD, for his contributions to the theory of automorphic functions; the Poncelet prize (2,000 francs) to Gen. P. CHARBONNIER, for his researches in ballistics; the Benjamin Valz prize (460 francs) to F. BOQUET, for the totality of his scientific work; the de Pontécoulant prize (700 francs) to Professor A. S. EDDINGTON, for his work in celestial mechanics; the Petit d'Ormoy prize (10,000 francs) to Professor H. LEBESGUE, for his contributions to mathematics. The foundation for scientific research has awarded the Gegner stipend (4,000 francs) to Professor R. BAIRE, for his work in the general theory of functions.

DR. J. FAIRON has been appointed professor of mathematics at the University of Liège.

PROFESSORS E. L. A. MERLIN and M. L. M. STUYVAERT have been promoted to full professorships of mathematics at the University of Ghent.

AT the University of Nancy, Dr. J. HAAG has been appointed professor of the calculus, and Professors GOT, of the Lycée at Marseilles, and LEAU, of the Lycée St. Louis, are in charge of courses. Professor A. M. G. FLOQUET has retired from active teaching.

ASSOCIATE professor E. GAU has been appointed to the professorship of infinitesimal analysis at the University of Grenoble, succeeding Professor COLLET, who has retired from active teaching.

AT French universities, the following maîtres de conférences have been appointed: MM. CHAPELON and KAMPÉ de FÉRIET, at the University of Lille; JANET, at the University of Grenoble; TURRIÈRE, at the University of Montpellier.

PROFESSOR V. J. BOUSSINESQ, of the University of Paris, has retired from active teaching.

DR. G. HESSENBERG has been appointed professor of mathematics at the University of Tübingen.

PROFESSOR W. KILLING, of the University of Munich, has retired from active teaching.

DR. F. SCHUR has been appointed to a professorship at the University of Breslau.

Dr. H. TIETZE has been appointed professor of mathematics at the University of Erlangen.

AT the University of Freiburg i. Br., Dr. A. LOEWY has been appointed professor of mathematics, and Professor L. STICKELBERGER has retired from active teaching.

THE library of the late Professor MAXIME BÔCHER, of Harvard, has been purchased for the Aberdeen Proving Ground.

THE mathematical library of the late Dr. R. A. HARRIS has been presented to Cornell University by Mrs. Harris. The collection comprises nearly three thousand volumes, including practically the complete literature on the theory of tides. It will be housed in the mathematical seminar.

AT the newly established southern division of the University of California, located at Los Angeles, the normal school and collegiate mathematics are the only branches of mathematics as yet provided. Miss MYRTIE COLLIER has been made head of the normal department, and Mr. G. E. F. SHERWOOD will have charge of the collegiate mathematics.

THE honorary degree of doctor of laws was conferred on Professor T. F. HOLGATE, of Northwestern University, by Queen's University, Ontario, in October.

ASSISTANT professor W. D. MACMILLAN, of the University of Chicago, has been promoted to an associate professorship of astronomy.

PROFESSOR A. A. BENNETT, of the University of Texas, has been granted a second year's leave of absence to serve as mathematics and dynamics expert in the ordnance bureau of the war department at Washington.

ASSISTANT professor H. N. DAVIS, of Harvard University, has been promoted to a full professorship of mechanical engineering.

AT the University of Michigan, the following changes have been made in the mathematical staff: associate professors PETER FIELD and L. C. KARPINSKI have been promoted to full professorships, and assistant professor J. W. BRADSHAW to an associate professorship; Dr. E. S. ALLEN has resigned to accept an assistant professorship at the University of West Virginia; Mr. E. C. BLANKENSTEIN, Mr. M. F. JOHNSON, Mr. D. C. KUZORINOFF, and Mr. O. J. PETERSON have been appointed instructors.

ASSISTANT professor W. E. MILNE, of the University of Oregon, has been promoted to a full professorship of mathematics.

AT the centennial celebration of Colgate University on October 11, JAMES M. TAYLOR, professor of mathematics in the University since 1870, received the honorary degree of doctor of science.

IN the mathematics department of the University of Minnesota, the following changes have been made: assistant professor R. R. SHUMWAY has been promoted to an associate professorship; Dr. C. H. YEATON has resigned to become professor of mathematics at the School of Engineering of Milwaukee; Miss OLIVE ATWOOD and Mr. R. M. MATHEWS have been appointed instructors.

PROFESSOR F. J. HOLDER, of the University of Pittsburgh, has been appointed professor of mathematics and dean of the school of commerce at Mercer University, Macon, Ga.

AT Vassar College, assistant professor LOUISE D. CUMMINGS has been promoted to an associate professorship and Dr. MARY E. WELLS to an assistant professorship of mathematics.

ASSOCIATE professor R. P. STEPHENS, of the University of Georgia, has been promoted to a full professorship of mathematics.

AT the Armour Institute of Technology, assistant professor W. C. KRATHWOHL has been promoted to an associate professorship and Dr. W. L. MISER, of the University of Arizona, has been appointed assistant professor of mathematics.

AT the University of California, associate professor C. A. NOBLE has been promoted to a full professorship of mathematics.

AT Washington University, St. Louis, assistant professor OTTO DUNKEL has been promoted to an associate professorship, and Dr. J. R. MUSSELMAN has been appointed instructor in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARWIN (A.). Ueber das Auflösen der Kongruenzen von dem dritten und vierten Grade nach einem Primzahlmodulus. Leipzig, O. Harrassowitz, 1916.

BARRAU (J. A.). Analytische meetkunde. Eerste deel: Het plat vlak. Groningen, Noordhoff, 1918. 8vo. 408 pp. Fl. 8.50

BIRKELAND (R.). Lærebok i matematisk analyse, differential- og integralregning. Differentialligninger. Trondhjem, F. Bruns, 1917. 8vo. 12 + 500 pp. Kr. 22.50

BOOLE (M. E.). Philosophy and fun of algebra. London, C. W. Daniel, 1918. Cr. 8vo. 3s.

CRATHORNE (A. R.). See RIETZ (H. L.).

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. 2ter Band. 4te Auflage. Leipzig, Teubner, 1919. 12 + 599 pp.

DAVISON (C.). Differential calculus for colleges and secondary schools. London, Bell, 1919. 8 + 309 pp. 6s.

DECKERT (A.). Einführung in die Vektor-Rechnung. (Sammlung Kösel.) Kempten und München, J. Kösel, 1916. 8vo. 4 + 84 pp. M. 1.95

FLEISCHMANN (K.). Die geodätischen Linien auf Rotationsflächen. (Diss.) Breslau, 1915.

HENRY (V.). Das erkenntnistheoretische Raumproblem in seinem gegenwärtigen Stande. Berlin, Reuther und Reichard, 1915. 8vo. 98 pp. M. 3.20

OTTO (F. A.). Herausforderung an alle Mathematiker der Welt oder die Lösung der Fermatschen Probleme. Berlin, C. Kroll, 1918. 8vo. 96 pp. M. 5.00

PINCHERLE (S.). Algebra complementare. 1a parte: Analisi algebrica. 3a edizione. (Manuali Hoepli, No. 141.) Milano, Hoepli, 1917. 8vo. 174 pp. L. 1.50

RIETZ (H. L.). and CRATHORNE (A. R.). College algebra. Revised edition. New York, Holt, 1919. 8vo. 12 + 268 pp.

SILBERSTEIN (L.). Projective vector algebra: an algebra of vectors independent of the axioms of congruence and of parallels. London, Bell, 1919. 7 + 78 pp. 7s. 6d.

STONEY (J.). An introduction to the differential and integral calculus, for the use of engineering and technical students. London, Pitman, 1918. Cr. 8vo. 146 pp. 3s. 6d.

II. ELEMENTARY MATHEMATICS.

AHRENS (W.). Mathematische Unterhaltungen und Spiele. 2ter Band. 2te Auflage. Leipzig, Teubner, 1918. 455 pp. Geb. M. 15.00

BETZ (W.). Geometry for junior high schools. (Elements of mathematics, book 1.) Rochester, N. Y., Board of Education, 1919. 12mo. 16 + 111 pp.

———. Introductory algebra exercises. Rochester, N. Y., Board of Education, 1918. 12mo. 6 + 73 pp.

Dennis (T.). An arithmetic for preparatory schools, with answers. 2d edition. London, Bell, 1919. 14 + 376 pp. 4s. 6d.

Junior geometry for schools. 5th edition. Dublin, Thom, 1918. 186 pp. 2s.

Milne (R. M.). Mathematical papers for admission into the Royal military academy and the Royal military college and papers in elementary engineering for naval cadetships for the years 1909–1918. Edited by R. M. Milne. London, Macmillan, 1919. 7s.

III. APPLIED MATHEMATICS.

Alger (P. L.). and Ingalls (J. M.). Groundwork of practical naval gunnery, or exterior ballistics. Annapolis, U. S. Naval Institute, 1918. 4to. 360 pp. $4.50

Auerbach (F.). Grundbegriffe der modernen Naturlehre. (Einführung in die Physik.) 4te Auflage. (Aus Natur und Geisteswelt, Nr. 40.) Leipzig, Teubner, 1917. 146 pp. Geb. M. 1.90

Bates (E. L.) and Charlesworth (F.). Practical mathematics. 2d edition. London, Batsford, 1918. Cr. 8vo. 5s.

Becq (A.). Electricité théorique et industrielle. Livre 1: Electricité théorique. Livre 2: Machines électriques. 8e édition. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1918. 8vo. 172 + 155 pp. Fr. 8.00 + 12.00

Bryk (O. J.). See Kepler (J.).

Card (S. F.). Air navigation: notes and examples. 3d edition. London, Arnold, 1919. 6 + 140 pp. 10s. 6d.

Charlesworth (F.). See Bates (E. L.).

Cyclopedia of drawing: a practical reference work on mechanical and architectural drawing; prepared by a staff of engineers, architects, designers, . . . 3d edition. Chicago, American Technical Society, 1919. 8vo. 1620 pp. 4 vols. $13.80

Dander (M.). Dictionnaire international de navigation aérienne et constructions aéronautiques (français, italien, anglais, allemand). Milano, Hoepli, 1919. 16mo. 227 pp. L. 6.50

Einstein (A.). Ueber die spezielle und die allgemeine Relativitätstheorie, gemeinverständlich. (Sammlung Vieweg, Tagesfragen aus den Gebieten der Naturwissenschaften und der Technik.) Braunschweig, Vieweg, 1917. 8vo. 4 + 76 pp. Geh. M. 2.80

Fry (H. P.). Notes on mechanical drawing. 6th edition. Philadelphia, H. P. Fry, 1919. 8vo. 85 pp. $1.50

Heuvelink (H. J.). De stereografische kaartprojectie in hare toepassing bij de rijksdriehoeksmeting. Delft, J. Waltman, Jr., 1918. 8vo. 48 pp.

Hicks (J. W.). The theory of the rifle and rifle shooting. London, Charles Griffin, 1919. 129 pp. 5s.

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HOHENNER (H.). Der Hohennersche Präzisionsdistanzmesser und seine Verbindung mit einem Theodolit. Leipzig, Teubner, 1919. 64 pp. Geh. M. 3.20

HOSMER (G. L.). Geodesy. Including astronomical observations, gravity measurements and method of least squares. New York, Wiley, 1919. 11 + 368 pp. $3.50

INGALLS (J. M.). See ALGER (P. L.).

JACOBY (H.). Navigation. 2d edition. New York, Macmillan, 1919. 11 + 350 pp.

JÄGER (G.). Theoretische Physik. I: Mechanik und Akustik. 4te verbesserte Auflage. (Sammlung Göschen, Nr. 76.) Berlin, Göschen, 1916. 16mo. 167 pp. Geb. M. 1.25

KEPLER (J.). Die Zusammenklänge der Welten. Neue Sternkunde. Auseinandersetzung mit dem Sternenherold. Schöpfungsgeheimnis in Weltentiefen. Herausgegeben von O. J. Bryk. Jena, Diederichs, 1918. 368 pp. Geh. M. 12.00

LEPPER (G. H.). From nebula to nebula or the dynamics of the heavens. 4th edition, revised and enlarged. Pittsburgh, G. H. Lepper, 1919. 401 pp.

LOCHE (L. E.). Des mécanismes élémentaires. Paris, 1919. 8vo. 12 + 257 pp. Fr. 12.00

LOPEZ SOLER (J.). Gráfico de calculos JuLius. Madrid, 1918.

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